JACOB'S LADDERS, CONJUGATE INTEGRALS, EXTERNAL MEAN-VALUES AND OTHER PROPERTIES OF A MULTIPLY $\pi(T)$-AUTOCORRELATION OF THE FUNCTION $|\zeta\left(\frac{1}{2}+it\right)|^2$

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Abstract. In this paper we obtain a new class of transformation formulae (without an explicit presence of a derivative) for the integrals containing products of factors $|\zeta\left(\frac{1}{2}+it\right)|^2$ with respect to two components of a disconnected set on the critical line.

1. Introduction

1.1. In the work of reference [3] (comp. also [1] and [2]) we have introduced the following disconnected set

$$\Delta(n+1)=\Delta(n+1;T,U)=\bigcup_{k=0}^{n+1}[\varphi_1^k(T),\varphi_1^k(T+U)] \tag{1.1}$$

where

$$\begin{aligned} & y=\frac{1}{2}\varphi(t)=\varphi_1(t);\ \varphi_1^0(t)=t,\ \varphi_1^1(t)=\varphi_1(t), \\ & \varphi_1^2(t)=\varphi_1[\varphi_1(t)],\dots,\varphi_1^k(t)=\varphi_1[\varphi_1^{k-1}(t)],\dots,\ t\in[T,T+U], \end{aligned} \tag{1.2}$$

and $\varphi_1^k(t)$ stands for the k-th iteration of the Jacob's ladder

$$\varphi_1(t),\ t\geq T_0[\varphi_1].$$

The set (1.1) has the following properties

$$\begin{aligned} & t\sim\varphi_1^k(t),\ \varphi_1^k(T)\geq(1-\epsilon)T,\ k=0,1,\dots,n+1, \\ & \varphi_1^k(T+U)-\varphi_1^k(T)<\frac{1}{2n+5}\frac{T}{\ln T},\ k=1,\dots,n+1, \\ & \varphi_1^k(T)-\varphi_1^{k+1}(T+U)>0.18\times\frac{T}{\ln T},\ k=0,1,\dots,n, \\ & U\in\left(0,\frac{T}{\ln^2 T}\right], \end{aligned} \tag{1.3}$$

and, in the macroscopic domain, i. e. for

$$U\in\left[T^{1/3+\epsilon},\frac{T}{\ln^2 T}\right], \tag{1.4}$$

we have a more detailed information about the set (1.1), namely

$$\begin{aligned} & |[\varphi_1^k(T),\varphi_1^k(T+U)]|=\varphi_1^k(T+U)-\varphi_1^k(T)\sim U,\ k=1,\dots,n+1, \\ & \varphi_1^k(T)-\varphi_1^{k+1}(T+U)\sim(1-c)\frac{T}{\ln T},\ k=0,1,\dots,n, \end{aligned} \tag{1.5}$$

Key words and phrases. Riemann zeta-function.

where c is the Euler constant. We have that (see (1.3))

$$[\varphi_1^{n+1}(T),\varphi_1^{n+1}(T+U)]\prec\dots\prec[\varphi_1^1(T),\varphi_1^1(T+U)]\prec[T,T+U],$$

i. e. the segments are ordered from $[T,T+U]$ to the left.

Remark 1. The asymptotic behavior of the disconnected set (1.1) is as follows: if $T\to\infty$ then the components of this set recedes unboundedly each from other (see (1.3), (1.5)) and all together are receding to infinity. Hence, if $T\to\infty$ then the set (1.1) behaves as an one-dimensional Friedman-Hubble expanding universe.

1.2. Next, we have shown (see [3]) that for the weighted mean-value of the integral

$$\int_T^{T+U}\prod_{k=0}^n\left|\zeta\left(\frac12+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t,\ U\in\left(0,\frac{T}{\ln^2T}\right] \tag{1.6}$$

the following factorization formula

$$\begin{aligned} &g_{n+1}\frac1U\int_T^{T+U}\prod_{k=0}^n\left|\zeta\left(\frac12+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim \\ &\sim\prod_{l=1}^s g_l\frac1U\int_T^{T+U}\prod_{k=0}^{a_{j_l}-1}\left|\zeta\left(\frac12+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t,\ T\to\infty \end{aligned} \tag{1.7}$$

holds true for every fixed natural number n and for every proper partition (the partition $n+1=n+1$ is excluded)

$$n+1=a_{j_1}+a_{j_2}+\dots+a_{j_s},\ a_{j_l}\in[1,n],\ l=1,\dots,s,$$

and

$$\begin{aligned} g_l&=\frac{U}{\varphi_1^{a_{j_l}}(T+U)-\varphi_1^{a_{j_l}}(T)},\ l=1,\dots,s, \\ g_{n+1}&=\frac{U}{\varphi_1^{n+1}(T+U)-\varphi_1^{n+1}(T)}. \end{aligned}$$

1.3. Next, by [3], (6.5), $n+1\to k$, we have

$$t-\varphi_1^k\sim k(1-c)\pi(t),\ k=0,1,\dots,n$$

where $\pi(t)$ is the prime-counting function. Hence

$$\begin{aligned} &\frac12+i\varphi_1^k(t)=\frac12+it-i[t-\varphi_1^k(t)]\sim \\ &\sim\frac12+it-ik(1-c)\pi(t),\ k=0,1,\dots,n. \end{aligned} \tag{1.8}$$

Remark 2. By (1.8) the arguments in the product (1.6) performs some complicated oscillations around the sequence

$$\frac12+it-ik(1-c)\pi(t),\ k=0,1,\dots,n$$

of the lattice points. Based on this, the integral (1.6) represents the multiple (for $k\geq 2$) $\pi(t)$-autocorrelation of the function $|\zeta\left(\frac12+it\right)|^2$, i. e. we have certain type of the complicated nonlinear and nonlocal interaction of the function $|\zeta\left(\frac12+it\right)|^2$ with itself.

1.4. After this we turn back to the formula (1.7). This formula binds the corresponding set of integrals over the same segment $[T,T+U]$. However, the segment $[T,T+U]$ is only one component of the disconnected set $\Delta(n+1)$ (see (1.1)). This is the reason for the following.

Question. Is there some formula that binds the integral (1.6) with the integral of the type

$$\int_{\varphi_1^{p(n)}(T)}^{\varphi_1^{p(n)}(T+U)}\prod_{k=0}^{n}\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(u)\right)\right|^2\mathrm{d}u,\ 1\leq p(n)\leq n,$$

i. e. with the integral over the component

$$[\varphi_1^{p(n)}(T),\varphi_1^{p(n)}(T+U)]\neq[T,T+U].$$

2. The main formula and its structure

2.1. We obtain the following theorem in the direction of our Question

Theorem. For every disconnected set

$$\Delta(2l)=\Delta(2l;T,U)=\bigcup_{k=0}^{2l}[\varphi_1^k(T),\varphi_1^k(T+U)],\ l=1,\dots,L_0$$

where $L_0\in\mathbb{N}$ is an arbitrary fixed number, and for every

$$U\in\left(0,\frac{T}{\ln^2 T}\right]$$

the following asymptotic transformation formula

$$\begin{aligned}
&\int_{\varphi_1^l(T)}^{\varphi_1^l(T+U)}\prod_{k=0}^{l-1}\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(u_l)\right)\right|^2\mathrm{d}u_l\sim \\
&\sim\frac{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)}{\varphi_1^l(T+U)-\varphi_1^l(T)}\int_T^{T+U}\prod_{k=0}^{l-1}\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t,\ T\to\infty
\end{aligned}\tag{2.1}$$

holds true.

Remark 3. We call the integrals that are bind by the formula (2.1) *the conjugate integrals.*

Let

$$\frac{1}{2}+i\gamma,\ \frac{1}{2}+i\gamma',\quad \gamma<\gamma'$$

be consecutive zeros of the Riemann zeta-function lying on the critical line and $l=7,T=\gamma,U=\gamma'-\gamma$. Thus, for example, the following formula (see (2.1))

$$\begin{aligned}
&\int_{\varphi_1^7(\gamma)}^{\varphi_1^7(\gamma')}\prod_{k=0}^{6}\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(u_7)\right)\right|^2\mathrm{d}u_7\sim \\
&\sim\frac{\varphi_1^{14}(T+U)-\varphi_1^{14}(T)}{\varphi_1^7(T+U)-\varphi_1^7(T)}\int_\gamma^{\gamma'}\prod_{k=0}^{6}\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t,\ \gamma\to\infty
\end{aligned}\tag{2.2}$$

holds true.

Remark 4. Nor the formula (2.2) for seven factors and $U=\gamma'-\gamma$ is not accessible for the current methods in the theory of the Riemann zeta-function.

2.2. By the continuity of the function $\varphi_1^l(v)$ we have (see (2.1)) that if

$$u_l=\varphi_1^l(t),\ t\in[T,T+U]$$

then

$$\varphi_1^k(u_l)=\varphi_1^k[\varphi_1^l(t)]=\varphi_1^{k+l}(t)\in[\varphi_1^{k+l}(T),\varphi_1^{k+l}(T+U)].$$

Consequently,the product

$$\prod_{k=0}^{l-1}\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(u_l)\right)\right|^2$$

corresponds to the disconnected set

$$\bigcup_{k=l}^{2l-1}[\varphi_1^k(T),\varphi_1^k(T+U)]=\Delta(l,2l-1), \tag{2.3}$$

and similarly the product

$$\prod_{k=0}^{l-1}\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2$$

corresponds to the disconnected set

$$\bigcup_{k=0}^{l-1}[\varphi_1^k(T),\varphi_1^k(T+U)]=\Delta(0,l-1), \tag{2.4}$$

where the sets (2.3), (2.4) are subsets of the set $\Delta(2l)$.

Next (comp. (1.3)), we have

$$\rho\{[\varphi_1^k(T),\varphi_1^k(T+U)];[\varphi_1^{k+1}(T),\varphi_1^{k+1}(T+U)]\}>0.17\times\pi(T) \tag{2.5}$$

where ρ represents the distance of corresponding segments.

Remark 5. The formula (2.1) controls a *quasi-chaotic* behavior of the values of the function $|\zeta\left(\frac{1}{2}+it\right)|^2$ with respect to the disconnected set $\Delta(2l)$ in spite of big distances separating the components of the set $\Delta(2l)$ (see (2.5)).

3. Some external mean-values

3.1. Using the mean-value theorem on the left-hand side of (2.1) we obtain

$$\begin{aligned}&\frac{1}{U}\int_T^{T+U}\prod_{k=0}^{l-1}\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim\\ &\sim\frac{\{\varphi_1^l(T+U)-\varphi_1^l(T)\}^2}{\{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)\}U}\prod_{k=0}^{l-1}\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(\alpha_l)\right)\right|^2\end{aligned} \tag{3.1}$$

where (see the paragraph 2.2)

$$\alpha_l\in(\varphi_1^l(T),\varphi_1^l(T+U)),\ \alpha_l=\varphi_1^l(t_l),$$

i. e.

$$\varphi_1^k(\alpha_l)=\varphi_1^{k+l}(t_l)\in(\varphi_1^{k+l}(T),\varphi_1^{k+l}(T+U)). \tag{3.2}$$

Hence, by (3.1) and (3.2) we have the following

Corollary 1. There are the values

$$\tau_k = \tau_k(T,U,l) \in (\varphi_1^k(T), \varphi_1^k(T+U)),\ k = l, \dots, 2l-1$$

such that

$$\begin{split} & \frac{1}{U}\int_T^{T+U} \prod_{k=0}^{l-1} \left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2 \mathrm{d}t \sim \\ & \sim \frac{\{\varphi_1^l(T+U)-\varphi_1^l(T)\}^2}{\{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)\}U} \prod_{k=l}^{2l-1}\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|^2 \end{split} \tag{3.3}$$

where

$$U \in \left(0, \frac{T}{\ln^2 T}\right],\ l = 1, \dots, L_0,\ T \to \infty.$$

Remark 6. Since:

(a) the integral

$$\int_T^{T+U} \prod_{k=0}^{l-1} \left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2 \mathrm{d}t$$

corresponds to the disconnected set $\Delta(0, l-1)$, (see (2.4)),

(b) the product

$$\prod_{k=l}^{2l-1}\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|$$

corresponds to the disconnected set $\Delta(l, 2l-1)$, (see (2.3)),

(c) the sets $\Delta(0, l-1)$ and $\Delta(l, 2l-1)$ are separated by the big distance

$$\rho\{\Delta(0,l-1); \Delta(l,2l-1)\} > 0.17 \times \pi(T)$$

(see (2.3), (2.4)),

it is quite natural to call the right-hand side of the equation (3.3) *the external mean-value* of the integral on the left-hand side.

3.2. Next, by the similar way, we obtain the following

Corollary 2. There are the values

$$\tau_k = \tau_k(T,U,l) \in (\varphi_1^k(T), \varphi_1^k(T+U)),\ k = 0, 1, \dots, l-1$$

such that

$$\begin{split} & \frac{1}{\varphi_1^l(T+U)-\varphi_1^l(T)}\int_{\varphi_1^l(T)}^{\varphi_1^l(T+U)} \prod_{k=0}^{l-1} \left|\zeta\left(\frac{1}{2}+i\varphi_1^k(u_l)\right)\right|^2 \mathrm{d}u_l \sim \\ & \sim \frac{\{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)\}U}{\{\varphi_1^l(T+U)-\varphi_1^l(T)\}^2} \prod_{k=0}^{l-1}\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|^2, \end{split} \tag{3.4}$$

where

$$U \in \left(0, \frac{T}{\ln^2 T}\right],\ l = 1, \dots, L_0,\ T \to \infty.$$

Remark 7. The formula (3.4) gives us the second variant of the external mean-value theorem.

4. Other properties of the distribution of the values of $|\zeta\left(\frac{1}{2}+it\right)|$ with respect to the disconnected set $\Delta(2l)$

4.1. Similarly to (3.3), (3.4), we obtain the following formula

$$
\begin{aligned}
&\prod_{k=0}^{l}\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|\sim \\
&\sim \frac{\varphi_1^l(T+U)-\varphi_1^l(T)}{\sqrt{\{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)\}U}}\prod_{k=l}^{2l-1}\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|,
\end{aligned}
\tag{4.1}
$$

where

$$\tau_k\in(\varphi_1^k(T),\varphi_1^k(T+U)),\ k=0,1,\dots,2l-1.$$

Next, we obtain from (4.1) the following

Corollary 3.

$$
\begin{aligned}
&G_0^{l-1}\left[\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|\right]\sim \\
&\sim\left\{\frac{\varphi_1^l(T+U)-\varphi_1^l(T)}{\sqrt{\{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)\}U}}\right\}^{1/l} G_l^{2l-1}\left[\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|\right],\ T\to\infty
\end{aligned}
\tag{4.2}
$$

where the following symbols

$$
\begin{aligned}
&G_0^{l-1}\left[\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|\right]=\left\{\prod_{k=0}^{l-1}\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|\right\}^{1/l}, \\
&G_l^{2l-1}\left[\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|\right]=\left\{\prod_{k=l}^{2l-1}\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|\right\}^{1/l}
\end{aligned}
\tag{4.3}
$$

stand for the geometric means.

4.2. Since (see (4.3))

$$\frac{G_0^{l-1}}{G_l^{2l-1}}=\bar{G}_0^{l-1}\left[\frac{\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|}{\left|\zeta\left(\frac{1}{2}+i\tau_{k+l}\right)\right|}\right], \tag{4.4}$$

and we have for arithmetic and geometric means (for example)

$$\bar{x}_A\geq\bar{x}_G;\ \bar{x}_A=\frac{1}{n}\sum_{i=1}^n x_i,\ \bar{x}_G=\sqrt[n]{\prod_{i=1}^n x_i},\ x_i>0. \tag{4.5}$$

Then we obtain from (4.2)-(4.4) the formula

$$\bar{G}_0^{l-1}\left[\frac{\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|}{\left|\zeta\left(\frac{1}{2}+i\tau_{k+l}\right)\right|}\right]\sim\left\{\frac{\varphi_1^l(T+U)-\varphi_1^l(T)}{\sqrt{\{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)\}U}}\right\}^{1/l}=\Omega_l.$$

Next, from the inequality

$$\bar{G}_0^{l-1}>(1-\epsilon)\Omega_l,\ T\to\infty$$

we obtain that (see (4.5))

$$\frac{1}{l}\sum_{k=0}^{l-1}\frac{\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|}{\left|\zeta\left(\frac{1}{2}+i\tau_{k+l}\right)\right|} > (1-\epsilon)\Omega_l. \tag{4.6}$$

The numbers $(\tau_0,\tau_1,\dots,\tau_{l-1})$ may be ordered by $l!$-ways in the product

$$\prod_{k=0}^{l-1}\frac{\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|}{\left|\zeta\left(\frac{1}{2}+i\tau_{k+l}\right)\right|},$$

and the same holds for the sequence of numbers $(\tau_l,\dots,\tau_{2l-1})$. Therefore we have $(l!)^2$ inequalities of the type (4.6). In this sense we use the symbol

$$\left\{\sum_{k=0}^{l-1}\frac{\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|}{\left|\zeta\left(\frac{1}{2}+i\tau_{k+l}\right)\right|}\right\}_m,\ m=1,\dots,(l!)^2.$$

Hence, we obtain from (4.6) the following

Corollary 4. We have $(l!)^2$ inequalities

$$\frac{1}{l}\left\{\sum_{k=0}^{l-1}\frac{\left|\zeta\left(\frac{1}{2}+i\tau_k\right)\right|}{\left|\zeta\left(\frac{1}{2}+i\tau_{k+l}\right)\right|}\right\}_m > (1-\epsilon)\left\{\frac{\varphi_1^l(T+U)-\varphi_1^l(T)}{\sqrt{\{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)\}U}}\right\}^{1/l},$$

for $\tau_0,\tau_1,\dots,\tau_{2l-1}$, where

$$m=1,\dots,(l!)^2,\ l=1,\dots,L_0,\ U\in\left(0,\frac{T}{\ln^2 T}\right],\ l=1,\dots,L_0,\ T\to\infty.$$

Remark 8. There are certain multiplicative effects also in the genetics, among the polygenic systems, and consequently the geometric means is used there, see, for example, [4], pp. 336, 337. We also note that we have used the formula for multiplication of independent variables as a motivation for our paper [3] .

5. Remarks about essential influence of the Riemann hypothesis on the sequence $\{\varphi_1^k(T+U)-\varphi_1^k(T)\}_{k=1}^{L_0}$

5.1. Let us remind that in the macroscopic case (1.4) we have the asymptotic formula (see (1.5))

$$\varphi_1^k(T+U)-\varphi_1^k(T)\sim U,\ k=1,\dots,L_0. \tag{5.1}$$

In connection with (5.1) we ask the question: what is the influence of the Riemann hypothesis on measures of the segments

$$[\varphi_1(T),\varphi_1(T+U)]$$

in the case (comp. (1.4))

$$U\in(0,T^{1/3-\epsilon_0}], \tag{5.2}$$

for example, in the case $\epsilon_0=\frac{1}{12}$, i. e.

$$U\in(0,T^{1/4}].$$

First of all we have, on the Riemann hypothesis, that (see [5], p. 300)

$$\zeta\left(\frac{1}{2}+it\right)=\mathcal{O}\left(t^{\frac{A}{\ln\ln t}}\right),\ t\to\infty, \tag{5.3}$$

i. e.

$$\zeta\left(\frac{1}{2}+it\right)=\mathcal{O}\left(T^{\frac{A}{\ln\ln T}}\right),\ t\in[(1-\epsilon)T,T+U] \tag{5.4}$$

(comp. (1.3) and [3], (6.17)). Next we obtain for (5.2) from our formula (see [2], (2.5))

$$\begin{aligned} &\int_T^{T+V}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t\sim[\varphi_1(T+V)-\varphi_1(T)]\ln T, \\ &V\in\left(0,\frac{T}{\ln T}\right], \end{aligned}$$

by (5.4) that

$$\begin{aligned} &\varphi_1^1(T+U)-\varphi_1^1(T)=\mathcal{O}\left(\frac{U}{\ln T}T^{\frac{2A}{\ln\ln T}}\right), \\ &\varphi_1^2(T+U)-\varphi_1^2(T)=\mathcal{O}\left(\frac{U}{\ln^2 T}T^{2\frac{2A}{\ln\ln T}}\right), \\ &\vdots \\ &\varphi_1^{L_0}(T+U)-\varphi_1^{L_0}(T)=\mathcal{O}\left(\frac{U}{\ln^{L_0} T}T^{L_0\frac{2A}{\ln\ln T}}\right). \end{aligned} \tag{5.5}$$

Since

$$T^{L_0\frac{2A}{\ln\ln T}}=T^{\frac{2L_0A}{\sqrt{\ln\ln T}}\frac{1}{\sqrt{\ln\ln T}}}<T^{\frac{1}{\sqrt{\ln\ln T}}}, \tag{5.6}$$

then by (5.5), (5.6) we obtain the following

Remark 9. On the Riemann hypothesis the following estimates hold true

$$\begin{aligned} &U\in(0,T^{1/3-\epsilon}]\ \Rightarrow\ \varphi_1^k(T+U)-\varphi_1^k(T)=\mathcal{O}\left(UT^{\frac{1}{\sqrt{\ln\ln T}}}\right), \\ &k=1,\dots,L_0. \end{aligned} \tag{5.7}$$

For example, if $U=1$ then on Riemann hypothesis we have that

$$\varphi_1^k(T+1)-\varphi_1^k(T)=\mathcal{O}\left(UT^{\frac{1}{\sqrt{\ln\ln T}}}\right),\ k=1,\dots,L_0$$

either for

$$L_0=S=10^{10^{10^{34}}}$$

(S is the Skeewes' constant).

5.2. In the general case (with or without the Riemann hypothesis) we have (comp. (5.3), (5.4))

$$\zeta\left(\frac{1}{2}+it\right)=\mathcal{O}(t^{1/6-\epsilon})=\mathcal{O}(T^{1/6-\epsilon}),\ t\in[(1-\epsilon)T,T+U],\ T\to\infty,$$

and consequently we obtain (comp. (5.5))

$$\begin{aligned} &\varphi_1^1(T+1)-\varphi_1^1(T)=\mathcal{O}(T^{2(1/6-\epsilon)})=\mathcal{O}(T^{1/3-2\epsilon}), \\ &\varphi_1^2(T+1)-\varphi_1^2(T)=\mathcal{O}(T^{4(1/6-\epsilon)})=\mathcal{O}(T^{2/3-4\epsilon}). \end{aligned}$$

Remark 10. In the general case we are able to guarantee only that

$$\varphi_1^1(T+1)-\varphi_1^1(T)\in(0,T^{1/3-\epsilon_0}],\ \epsilon\leq\frac{\epsilon_0}{2}. \tag{5.8}$$

Hence, the comparison of (5.7), $U=1$, with (5.8) shows the essential influence of the Riemann hypothesis on our subject.

6. The proof of Theorem

6.1. By using our formula (see [2], (9.1))

$$\tilde{Z}^2(t)=\frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t}$$

we obtain (see (1.2))

$$\begin{aligned}
&\int_T^{T+U}\prod_{k=0}^n\tilde{Z}^2[\varphi_1^k(t)]\mathrm{d}t= \\
&=\int_T^{T+U}\tilde{Z}^2[\varphi_1^n(t)]\tilde{Z}^2[\varphi_1^{n-1}(t)]\cdots\tilde{Z}^2[\varphi_1^1(t)]\tilde{Z}^2[t]\mathrm{d}t= \\
&=\int_T^{T+U}\tilde{Z}^2[\varphi_1^{n-1}(\varphi_1^1(t))]\tilde{Z}^2[\varphi_1^{n-2}(\varphi_1^1(t))]\cdots\tilde{Z}^2[\varphi_1^1(t)]\frac{\mathrm{d}\varphi_1^1(t)}{\mathrm{d}t}\mathrm{d}t= \\
&=\int_{\varphi_1^1(T)}^{\varphi_1^1(T+U)}\tilde{Z}^2[\varphi_1^{n-1}(u_1)]\tilde{Z}^2[\varphi_1^{n-2}(u_1)]\cdots\tilde{Z}^2[\varphi_1^1(u_1)]\tilde{Z}^2[u_1]\mathrm{d}u_1= \\
&=\int_{\varphi_1^1(T)}^{\varphi_1^1(T+U)}\tilde{Z}^2[\varphi_1^{n-2}(\varphi_1^1(u_1))]\cdots\tilde{Z}^2[\varphi_1^1(u_1)]\frac{\mathrm{d}\varphi_1^1(u_1)}{\mathrm{d}u_1}\mathrm{d}u_1= \\
&=\int_{\varphi_1^2(T)}^{\varphi_1^2(T+U)}\tilde{Z}^2[\varphi_1^{n-2}(u_2)]\cdots\tilde{Z}^2[u_2]\mathrm{d}u_2=\cdots= \\
&=\int_{\varphi_1^l(T)}^{\varphi_1^l(T+U)}\tilde{Z}^2[\varphi_1^{n-l}(u_l)]\cdots\tilde{Z}^2[\varphi_1^0(u_l)]\mathrm{d}u_l,\ l=1,\dots,n,
\end{aligned}$$

i. e. the following formula

$$\begin{aligned}
&\int_T^{T+U}\prod_{k=0}^n\tilde{Z}^2[\varphi_1^k(t)]\mathrm{d}t=\int_{\varphi_1^l(T)}^{\varphi_1^l(T+U)}\prod_{k=0}^{n-l}\tilde{Z}^2[\varphi_1^k(u_l)]\mathrm{d}u_l, \\
&l=1,\dots,n
\end{aligned} \tag{6.1}$$

holds true.

6.2. Let us remind that (see [3], (6.14))

$$\begin{aligned}
&\tilde{Z}^2(t)=\frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]}=\frac{\left|\zeta\left(\frac12+it\right)\right|^2}{\left\{1+\mathcal{O}\left(\frac{\ln\ln T}{\ln T}\right)\right\}\ln t}, \\
&t\in[T,T+U],\ U\in\left(0,\frac{T}{\ln T}\right], \\
&(\varphi_1^l(T),\varphi_1^l(T+U))\subset(\varphi_1^{n+1}(T),T+U).
\end{aligned} \tag{6.2}$$

Putting (6.2) into (6.1) and using the mean-value theorem on both integrals in (6.1) we obtain the following formula (comp. [3], (6.17))

$$\begin{aligned}&\int_T^{T+U}\prod_{k=0}^n\left|\zeta\left(\frac 12+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim \\ &\sim \ln^l T\int_{\varphi_1^l(T)}^{\varphi_1^l(T+U)}\prod_{k=0}^{n-l}\left|\zeta\left(\frac 12+i\varphi_1^k(u)\right)\right|^2\mathrm{d}u,\ l=1,\dots,n,\ T\to\infty.\end{aligned}\tag{6.3}$$

Next, the formula (see [3], (3.1))

$$\begin{aligned}&\int_T^{T+U}\prod_{k=0}^n\left|\zeta\left(\frac 12+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim\{\varphi_1^{n+1}(T+U)-\varphi_1^{n+1}(T)\}\ln^{n+1}T; \\ &\ln^{n+1}T=\ln^{(l-1)+1}T\ln^{(n-l)+1}T\end{aligned}\tag{6.4}$$

together with the formula (6.3) gives the following asymptotic equality

$$\frac{\int_T^{T+U}}{\varphi_1^{n+1}(T+U)-\varphi_1^{n+1}(T)}\sim\frac{\int_T^{T+U}}{\int_{\varphi_1^l(T)}^{\varphi_1^l(T+U)}}\frac{\int_T^{T+U}}{\int_{\varphi_1^{n+1-l}(T)}^{\varphi_1^{n+1-l}(T+U)}},$$

i. e.

$$\begin{aligned}&\{\varphi_1^{n+1}(T+U)-\varphi_1^{n+1}(T)\}\int_T^{T+U}\prod_{k=0}^n\left|\zeta\left(\frac 12+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim \\ &\sim\int_{\varphi_1^l(T)}^{\varphi_1^l(T+U)}\prod_{k=0}^{n-l}\left|\zeta\left(\frac 12+i\varphi_1^k(u)\right)\right|^2\mathrm{d}u\times \\ &\times\int_{\varphi_1^{n+1-l}(T)}^{\varphi_1^{n+1-l}(T+U)}\prod_{k=0}^{l-1}\left|\zeta\left(\frac 12+i\varphi_1^k(v)\right)\right|^2\mathrm{d}v,\ l=1,\dots,n,\ T\to\infty.\end{aligned}\tag{6.5}$$

6.3. Next, in the case

$$n-l=l-1\ \Rightarrow\ n=2l-1,$$

we obtain that (see (6.4), (6.5))

$$\begin{aligned}&\left\{\int_{\varphi_1^l(T)}^{\varphi_1^l(T+U)}\prod_{k=0}^{l-1}\left|\zeta\left(\frac 12+i\varphi_1^k(u_l)\right)\right|^2\mathrm{d}u_l\right\}^2\sim \\ &\sim\{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)\}\int_T^{T+U}\prod_{k=0}^{2l-1}\left|\zeta\left(\frac 12+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim \\ &\sim\{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)\}^2\ln^{2l}T,\end{aligned}$$

i. e. the following formula holds true

$$\begin{aligned}&\int_{\varphi_1^l(T)}^{\varphi_1^l(T+U)}\prod_{k=0}^{l-1}\left|\zeta\left(\frac 12+i\varphi_1^k(u_l)\right)\right|^2\mathrm{d}u_l\sim \\ &\sim\{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)\}\ln^l T.\end{aligned}\tag{6.6}$$

Consequently, we obtain from (6.6) by (6.4), in the case $n=l-1$, the formula

$$\int_{\varphi_1^l(T)}^{\varphi_1^l(T+U)}\prod_{k=0}^{l-1}\left|\zeta\left(\frac 12+i\varphi_1^k(u_l)\right)\right|^2\mathrm{d}u_l\sim$$
$$\sim\frac{\varphi_1^{2l}(T+U)-\varphi_1^{2l}(T)}{\varphi_1^l(T+U)-\varphi_1^l(T)}\int_T^{T+U}\prod_{k=0}^{l-1}\left|\zeta\left(\frac 12+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t$$

that verifies (2.1).

I would like to thank Michal Demetrian for his help with electronic version of this paper.

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